Analytical results of a bulk-service queue with size-dependent service time

Dissertation submitted to
Visvesvaraya National Institute of Technology, Nagpur
in partial fulfillment of the requirement for the award of the degree

Master of Science

in

Mathematics

by

Aditya Yadav
(MS23MTH013)

under the supervision of

Dr. Sourav Pradhan

Department of Mathematics
Visvesvaraya National Institute of Technology
Nagpur - 440010 (India)
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Declaration

I, Aditya Yadav, hereby declare that this dissertation titled "Analytical results of a bulk-service queue with size-dependent service time" is carried out by me at the Department of Mathematics of Visvesvaraya National Institute of Technology, Nagpur. The work is original and has not been submitted earlier whole or in part for the award of any degree/diploma at this or any other institution/University.

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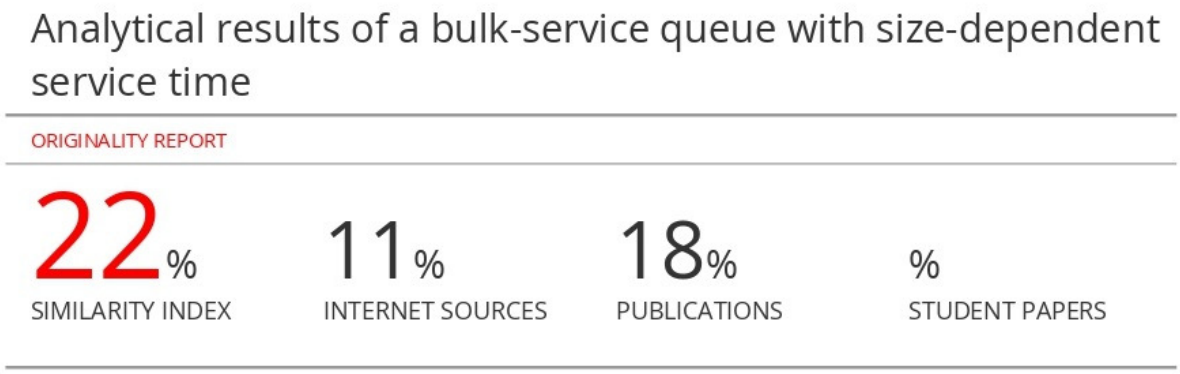

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Acknowledgement

It is with deep sense of gratitude that I express my sincere thanks to my supervisor, Dr. Sourav Pradhan, Assistant Professor, Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur, for his invaluable guidance, encouragement, and constant support throughout my project work. Without his active mentorship, this work would not have been possible. His profound academic insight, along with his generous support both academically and personally, are deeply appreciated.

I would also like to extend my heartfelt thanks to Dr. Pallavi Mahale, Professor and Head of the Department of Mathematics, Dr. P. P. Chakravarthy, Professor, and Dr. Neelima Bhengra, Assistant Professor, Department of Mathematics, VNIT Nagpur, for providing excellent facilities and for their continued encouragement during the course of this project. I am grateful to the Project Assessment and Progress Committee members, especially Dr. M. Devakar, Professor, Department of Mathematics, VNIT Nagpur, for their valuable suggestions and support.

I sincerely appreciate the assistance and advice of Mr. Prasenjit Karan, Ph.D. research scholar, Mr. Satyam Das, Ph.D. research scholar, Mr. Ashish Verma, Ph.D. research scholar, Department of Mathematics, VNIT Nagpur, for their time, helpful discussions, and valuable recommendations.

I am forever indebted to my parents and family members for their unconditional love, blessings, and constant encouragement.Finally, I bow in reverence to the Almighty, whose blessings have enabled me to successfully complete this work.

Aditya Yadav

Abstract

Batch-service queues with service times that depend on the batch size are widely used in various practical settings such as group testing of blood samples for detecting any kind of diseases. Our primary aim is to determine the distribution of the number of customers in the queue by analyzing an infinite-buffer batch-service queueing model where the service time depends on the batch size and follows general service time distribution with general bulk service (a,b) rule. To analyze the model, we employ the embedded Markov chain technique by constructing the transition probability matrix of the underlying system. Using the ergodicity property, we derive the probability generating function (pgf) of the model. Additionally, we apply the partial fraction method, which is commonly used to extract probabilities from the pgf.

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Symbols/Abbreviations	Name
TPM	transition probability matrix
pgf	probability generating function
$\Lambda^{+}(z)$	pgf of number of customers in the queue at departure epoch
α_n^{+}	probability that n customers are in the queue at departure epoch
$h_i^{(r)}$	probability of j customer arrivals during service time of a batch of size r

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Chapter 1

Introduction

1.1 Brief Discussion of Queues

Queues are generally known as waiting lines. In a queueing system the customers arrive and wait for their turn to get required services, then it takes the service and leave the system. Thus it is an inescapable and well-known phenomenon which occurs in our daily life. For example, (i) cars waiting at toll booths, (ii) customers waiting in a line at supermarkets to check out, (iii) people have to wait in a bank to withdraw/deposit money. It is often seen that due to long waiting time in queue the customers leave the system, thus causing loss to the companies. Then we can say that customers as well as service providers are not very happy due to long queue. Queues can not be avoided, but waiting time of a customer can be reduced by a proper scientific planning. Queueing theory is mathematical analysis of waiting lines. A queuing system can be described as customers arriving from a finite/infinite population, waiting in a finite/infinite line for service, (if it is not immediate) and leaving the system after being served.

Firstly, we briefly discuss the components of a queuing system. The abbreviated form of a queuing model is denoted by **A/B/C/D/E/F**, where

- **A** denotes the arrival pattern. It could be in accordance with Poisson process or any general distribution.
- **B** describes the service time distribution.
- **C** denotes the number of server which can be single or multiple.
- **D** indicates the system capacity which may be finite or infinite.

- **E** indicates the source capacity from which the customer arrives.
- **F** indicates the queue discipline i.e., the order in which customers are served. For example first come first served (FCFS), last come first served (LCFS), priorities etc. An example of priority queue discipline is VIP clients or customers with premium memberships often receive priority service over regular customers.

As an example of queueing system, we can have $M/G/1/\infty/\infty/FCFS$, where M stands for Poisson arrival process, service time follows general distribution, there is a single server, infinite waiting capacity, infinite source population size and first come first serve.

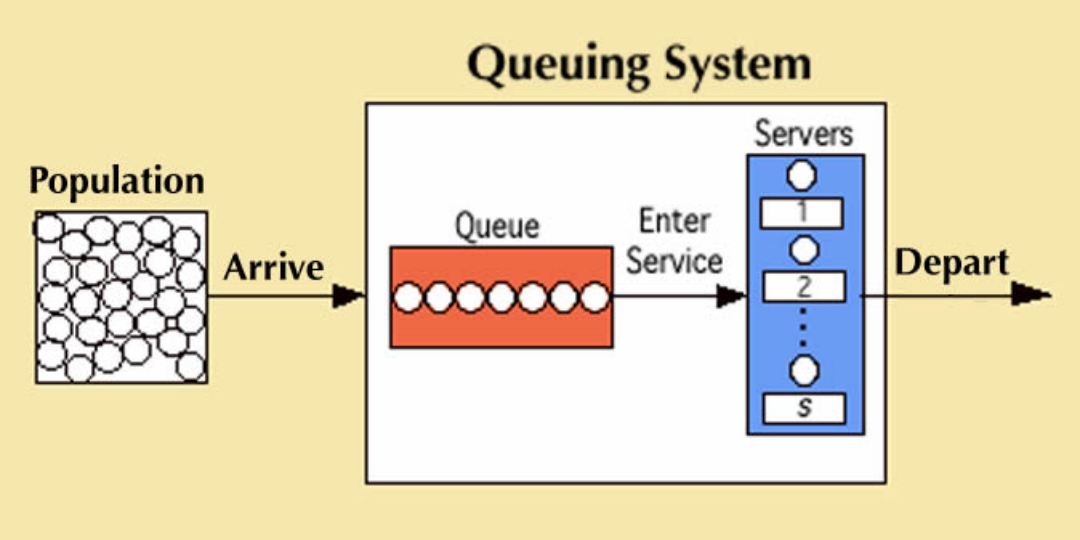

Figure 1.1: Queuing System

After a brief and short introduction about queueing models, we now concentrate on different type of queues. In recent times, researchers are quiet interested to opt queueing theory as their research area. Existing literatures include both theoretical and computational aspects of the queueing models. The basics of queueing theory has been discussed in the books by Medhi [1], Gross et al. [2], Chaudhry and Templeton [3].

1.2 Different Type of Queues

- **Batch arrival and batch service queue:** A batch arrival queue is a type of queuing system where multiple items or customers arrive at the same time as a single group or batch. The number of items in each arriving batch can be fixed or variable.

  A batch service queue is a type of queuing system where the server processes multiple

items or customers simultaneously as a single batch. The server processes a batch of items together, and the service time can depend on the batch size or be fixed for all batches. The number of items served in each batch can be fixed or variable.

- **Retrial Queue:** Suppose that an arriving customers finds the server is busy, then the customer may leave the system temporarily and move to some virtual place called as 'orbit' and after some certain random time they repeat their request. This type of queues are generally called as retrial queues and have been mainly used to solve problems which arise in telecommunication, telephone switching, web access, computer systems, etc..

- **Vacation Queue:** Whenever the system is empty, server leave the parent queue and perform some auxiliary works. This specified duration is generally known as vacation. Various vacation policies such as single vacation, multiple vacations, N-limited scheme, differentiated vacation, working vacation etc. have been proposed by several authors. However, in order to increase the energy efficiency telecommunication networks, the popular strategies are single and multiple vacation which makes the analysis more realistic and flexible. In the performance analysis of computer and manufacturing systems, server vacation are considered to serve as effective instruments. There exists another type of vacation period known as working vacation in which server continues to carry out the process with a slower rate instead of stopping completely. Such type of queues are applicable in the modeling analysis of transportation system, manufacturing and production engineering, and so forth.

- **Queues with optional service:** n several real-life situations, one may encounter with the queues where all the arriving customers require the essential service first and then some of them may require supplementary/extra service. For a classic example, all the ships arriving at a port need unloading of items (which is considered in essential service) but only some of them may require re-loading soon after the unloading which is processed as additional/optional service. Such practical problems can be modeled as queues with optional service.

1.3 Motivation and Objectives

In real-life cases batch-service queues plays a appreciable role while decisive information is required and also increase the server efficiency. The cost of the system can be reduced by giving

the service in batches instead of individually. Specifically batch-size-dependent service queues have been paid more attention by researchers in recent times due to its outspread applications.

Our objectives are:

- Firstly, to provide mathematical formulation of the queue under consideration.
- Then, to construct the transition probability matrix at departure epoch.
- To derive the probability generating function of queue length distribution in steady-state, from which to extract the complete distribution using the roots of the associated characteristic equation.
- Our objective also extends to obtain average number of customers in the queue and other performance measures from the known distribution.

Chapter 2

Results of a group-service queue with general distribution

2.1 Introduction

In the previous chapter, we have provided a comprehensive introduction to queueing theory, which clearly outlines the motivation behind the primary project work. The main emphasis of this chapter is to present the main work done, which involves the modeling of an infinite-buffer batch-service queue where the service time is contingent upon the batch size being serviced. We begin by detailing the model thoroughly. Next, we aim to construct the transition probability matrix (TPM). We derive the probability generating function (pgf) for the queue length distribution at the departure epoch. Subsequently, we must evaluate the unknown probabilities present in the pgf before proceeding to extract all probabilities using the partial fraction method. Ultimately, we derive the average queue length from the pgf. The application of this particular queueing model is well discussed by Barlev et al. [4].

2.2 Model Description

This section exhibits the components involved in the model under study.

- The concerned queue is abbreviated as $M/G_n^{(a,b)}/1$.
- **Arrival process:** Generally, we need an arrival process which is analytically tractable and free from any fractal features such as self similarity. Hence, we assume the popular Poisson arrival process with rate λ.

- **Service rule:** Batch-service rule is general bulk-service (a,b) rule. i.e., the server initiates the service with a minimum number of customers say, 'a' $(a \geq 1)$, and can provide the service maximum number of customers 'b'. All 'r' the customers get service together if $a \leq r \leq b$.

- **Service process:** In this case, result is developed by employing general distribution with density function $v_r(t),\ t \geq 0,\ a \leq r \leq b$, Laplace transform $V_r^*(\theta)$, and mean $\dfrac{1}{\mu_r}$.

- **Traffic intensity:** The traffic intensity of the concerned model is $\rho = \frac{\lambda}{b\,\mu_b} < 1$.

- Here, the system consists of single server, infinite waiting capacity, infinite source capacity, and FCFS.

2.3 Construction of Transition Probability Matrix

After describing the model , we now focus to construct the TPM by developing a recurrence relation, given by:

$$N_{n+1}^{+} = \begin{cases} B_{n+1}^{(a)}, & \text{if } N^{+} = 0,1,2,3,\cdots,a \\ B_{n+1}^{(a+1)}, & \text{if } N^{+} = a+1 \\ \vdots & \vdots \\ B_{n+1}^{(b)}, & \text{if } N_n^{+} = b \\ N_n^{+} - b + B_{n+1}^{(b)}, & \text{if } N_n^{+} > b \end{cases}$$

where N_{n+1}^{+} represent the number of customers waiting in the queue after the departure epoch of the $(n+1)$th batch, $B_{n+1}^{(r)}$ is the number of customers that arrive during the service time of $(n+1)$th batch of size r. Employing the probability distribution of $B_{n+1}^{(r)}$, we achieve the matrix with each element being one step transition probability.

$$
P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ a-1 \\ a \\ a+1 \\ \vdots \\ B \\ B+1 \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & \dots & \end{array} \\
\begin{bmatrix}
h_0^{(a)} & h_1^{(a)} & h_2^{(a)} & h_3^{(a)} & \cdot & \cdot & \cdot \\
h_0^{(a)} & h_1^{(a)} & h_2^{(a)} & h_3^{(a)} & \cdot & \cdot & \cdot \\
h_0^{(a)} & h_1^{(a)} & h_2^{(a)} & h_3^{(a)} & \cdot & \cdot & \cdot \\
\vdots & \vdots & \vdots & \vdots & & & \\
h_0^{(a)} & h_1^{(a)} & h_2^{(a)} & h_3^{(a)} & \cdot & \cdot & \cdot \\
h_0^{(a)} & h_1^{(a)} & h_2^{(a)} & h_3^{(a)} & \cdot & \cdot & \cdot \\
h_0^{(a+1)} & h_1^{(a+1)} & h_2^{(a+1)} & h_3^{(a+1)} & \cdot & \cdot & \cdot \\
\vdots & \vdots & \vdots & \vdots & & & \\
h_0^{(b)} & h_1^{(b)} & h_2^{(b)} & h_3^{(b)} & \cdot & \cdot & \cdot \\
0 & h_0^{(b)} & h_1^{(b)} & h_2^{(b)} & \cdot & \cdot & \cdot \\
\vdots & \vdots & \vdots & \vdots & & &
\end{bmatrix}
\end{array}
$$

where $h_i^{(r)}$, $a \le r \le b$ designates the probability of i arrival during the processing time of a batch of size r. The corresponding pgf is

$$H^{(r)}(z) = \sum_{i=0}^{\infty} h_i^{(r)} z^i = V_r^*(\lambda - \lambda z).$$

2.4 Derivation of the Pgf

After having the TPM ,we now concentrate on the derivation of queue length at departure epoch. Each state of the system being ergodic, the stationary distribution exists andindependent of intial probability. The steady-state probabilities are the solution of the equations $\alpha^+ P = \alpha^+$, with $\alpha^+ e = 1$, where e stand for column vector with each element as 1, and $\alpha^+ = [\alpha_0^+, \alpha_1^+, \alpha_2^+, \cdots]$. The corresponding pgf is $\Lambda^+(z) = \sum_{i=0}^{\infty} \alpha_i^+ z^i$ determination of which is our concern. We are led to the following system of equations:

$$\alpha_0^+ = \sum_{j=0}^{a} h_0^{(a)} \alpha_j^+ + \sum_{j=a+1}^{b} h_0^{(j)} \alpha_j^+ \tag{2.1}$$

$$\alpha_1^+ = \sum_{j=0}^{a} h_1^{(a)} \alpha_j^+ + \sum_{j=a+1}^{b} h_1^{(j)} \alpha_j^+ + h_0^{(b)} \alpha_{b+1}^+ \tag{2.2}$$

$$\alpha_2^+ = \sum_{j=0}^{a} h_2^{(a)} \alpha_j^+ + \sum_{j=a+1}^{b} h_2^{(j)} + {}_j + \sum_{j=b+1}^{b+2} h_{i-j+b}^{(b)} \alpha_j^+ \tag{2.3}$$

$$\cdots$$

In general, we obtain

$$\alpha_i^+ = \sum_{j=0}^{a} h_i^{(a)} \alpha_j^+ + \sum_{j=a+1}^{b} h_i^{(j)} \alpha_j^+ + \sum_{j=b+1}^{b+i} h_{i-j+b}^{(b)} \alpha_j^+, \quad i \geq 1 \tag{2.4}$$

Multiplying (2.1) by z^0, (2.2) by z^1, (2.3) by z^2 and so on, we obtain

$$\begin{aligned}
\sum_{i=0}^{\infty} \alpha_i^+ z^i &= \sum_{i=0}^{\infty} \sum_{j=0}^{a} h_i^{(a)} \alpha_j^+ z^i + \sum_{i=0}^{\infty} \sum_{j=a+1}^{b} h_i^{(j)} \alpha_j^+ z^i + \sum_{i=0}^{\infty} \sum_{j=b+1}^{b+i} h_{i-j+b}^{(b)} \alpha_j^+ z^i \\
\Rightarrow \Lambda^+(z) &= \sum_{j=0}^{a} \alpha_j^+ H^{(a)}(z) + \sum_{j=a+1}^{b} \alpha_j^+ H^{(j)}(z) + \sum_{j=b+1}^{\infty} \alpha_j^+ z^{j-b} \sum_{i=j-b}^{\infty} h_{i-j+b}^{(b)} z^{i-j+b} \\
\Rightarrow \Lambda^+(z) &= \sum_{j=0}^{a} \alpha_j^+ H^{(a)}(z) + \sum_{j=a+1}^{b} \alpha_j^+ H^{(j)}(z) + \sum_{j=b+1}^{\infty} \alpha_j^+ z^{j-b} H^{(b)}(z) \\
\Rightarrow \Lambda^+(z) &= \sum_{j=0}^{a} \alpha_j^+ H^{(a)}(z) + \sum_{j=a+1}^{b} \alpha_j^+ H^{(a)}(z) + \frac{H^{(b)}(z)}{z^b} \left[\sum_{j=0}^{\infty} \alpha_j^+ z^j - \sum_{j=0}^{b} \alpha_j^+ z^j \right]
\end{aligned}$$

After some algebraic simplification, we get the pgf

$$\Lambda^+(z) = \frac{\sum_{n=0}^{a} \alpha_n^+ \left[z^b H^{(a)}(z) - z^n H^{(b)}(z) \right] - \sum_{n=a+1}^{b-1} \alpha_n^+ \left[H^{(n)}(z) z^b - z^n H^{(b)}(z) \right]}{z^b - H^{(b)}(z)} \tag{2.5}$$

The pgf is completely known to us except b unknown probabilities $\alpha_0^+, \alpha_1^+, \alpha_2^+, \cdots \alpha_{b-1}^+$ appering in the numerator of the pgf.

2.5 Determination of Unknown Probabilities

In equation (2.5), numerator of pgf contains b unknown probabilities: $\alpha_0^+, \alpha_1^+, \alpha_2^+, \cdots \alpha_{b-1}^+$. Firstly we find the unknown probabilities with the help of well known Rouche's theorem by which it can be concluded that the denominator $z^b - H^{(b)}(z)$ has exactly b zeros in $|z| \leq 1$. In order to prove this, first let $f(z) = z^b$ and $g(z) = -H^{(b)}(z)$. On the unit circle, we have $|z| = 1+\varepsilon$ in the complex plane, for small $\varepsilon > 0$.

$$\begin{aligned}
|g(z)| &= |H^{(b)}(z)| \\
&\leq H^{(b)}(|z|) \\
&= H^{(b)}(1+\varepsilon)
\end{aligned}$$

$$
\begin{aligned}
&= H^{(b)}(1)+\varepsilon H^{(b)'}(1)+\frac{\varepsilon^2}{2!}H^{(b)''}(1)+... \\
&= 1+\varepsilon b\rho+o(\varepsilon)\ as\ H^{(b)'}(1)=b\rho \\
&\leq 1+\varepsilon b\ \ as\ \rho<1 \\
&\leq (1+\varepsilon)^b=|z|^b \\
&= |f(z)|.
\end{aligned}
$$

Using Rouche's theorem, one can conclude that $f(z)+g(z)$ has equal number of zeros as $f(z)$ on $|z|=1+\varepsilon$. As $f(z)=z^b$ possesses b zeros and when $\varepsilon\to 0$, we find that $f(z)+g(z)=z^b-H^{(b)}(z)$ possesses exactly b zeros in $|z|\leq 1$.

Now we turn your focus to determine those unknowns. Since the pgf is a analytic function in $|z|<1$, the numerator has to vanish at $z=z_1,\ z_2,\ z_3,\cdots\ z_{b-1}$. This leads to

$$
\sum_{n=0}^{a}\alpha_n^+\left[z_i^bH^{(a)}(z_i)-z_i^nH^{(b)}(z_i)\right]-\sum_{n=a+1}^{b-1}\alpha_n^+\left[H^{(n)}(z_i)z_i^b-z_i^nH^{(b)}(z_i)\right]=0,\quad i=1,2,\cdots b-1.
$$

It may be noted here that for $z_b=1$ any equations will not be generated. As we need to find b unknowns $\alpha_0^+,\alpha_1^+,\alpha_2^+,\cdots\alpha_{b-1}^+$, we need one more equation. This can be accomplished using the normalizing conditions $\Lambda^+(1)=1$. Applying L-Hospital rule, and using $H^{(b)}(1)=1$and $H^{(b)'}(1)=b\rho$, we get

$$
\sum_{n=0}^{a}\alpha_n^+\left[(b-n)+\lambda\left(\frac{1}{\mu_a}-\frac{1}{\mu_b}\right)\right]+\sum_{n=a+1}^{b-1}\alpha_n^+\left[(b-n)+\lambda\left(\frac{1}{\mu_a}-\frac{1}{\mu_b}\right)\right]=(b-b\rho). \quad (2.6)
$$

Now the unknown probabilities $\alpha_0^+,\alpha_1^+,\alpha_2^+,\cdots\alpha_{b-1}^+$ can be determined uniquely by solving above system of equations.

Remark: From this queueing system, if we substitute $H^{(n)}(z)=H(z)$ for all $a\leq n\leq b$, then we get the pgf of $M/G^{(a,b)}/1$ queue without batch-size-dependent service. This result exactly match with Choudhry and Templeton [3].

2.6 Extraction of Probabilities $(\alpha_n^+)_0^\infty$ From the Pgf

After determination the b unknown probabilities α_n^+'s $(0 \leq n \leq b-1)$, $\Lambda^+(z)$ is entirely known to us. Now our main objectives are to get all the state probabilities α_n^+, $n \geq 0$. As both the numerator (say $N_1(z)$) and denominator (say$D_1(z)$) are known polynomial in z, we can apply partial-fraction expansion.In order to do this we have the knowledge about all zeros of $D_1(z)$. Let $N_1(z)$ and $D_1(z)$ are the polynomials of degree L_1 and M_1, respectively. The distribution of service time, service and batch size plays the role in deciding the degree of $N_1(z)$ and $D_1(z)$.
Now we obtain the probability distributions after discussing the case only for distinct roots.
It is already discussed that $N_1(z)$ has b zeros on $|z| \leq 1$. Consequently, $D_1(z)$ must contain $M_1 - b$ distinct zeros in$|z| > 1$. Let us call these zeros as $\gamma_1, \gamma_2, \gamma_3, \cdots \gamma_{M_1-b}$.

Case 1: $L_1 \geq M_1$

Using the partial-fraction expansion, we can uniquely write the rational function $\Lambda^+(z)$ as

$$\Lambda^+(z) = \sum_{j=0}^{L_1-M_1} \beta_j z^j + \sum_{j=1}^{M_1-b} \frac{d_k}{\gamma_j - z} \tag{2.7}$$

for some constant β_j and d_k. The first sum stands for the resultant of division of $N_1(z)$ by $D_1(z)$. Employment of residue theorem leads to

$$d_j = -\frac{N_1'(\gamma_j)}{D_1'(\gamma_j)}, \quad j = 1,2,3,...,M_1 - b.$$

Collection of the coefficient of z^n from above give rise to

$$\alpha_n^+ = \beta_n + \sum_{j=1}^{M_1-b} \frac{d_j}{\gamma_j^{n+1}}, \ n \geq 0. \tag{2.8}$$

Case 2: $L_1 < M_1$

After applying partial friction technique on $\Lambda^+(z)$, we have

$$\Lambda^+(z) = \sum_{j=1}^{M_1-b} \frac{d_j}{\gamma_j - z} \tag{2.9}$$

where

$$d_j = -\frac{N_1'(\gamma_j)}{D_1'(\gamma_j)}, \quad j = 1,2,3,...,M_1 - b.$$

Now, one can collect the coefficient of z^n from both sides of in order to have the distribution and

are given by

$$\alpha_n^+ = \sum_{j=1}^{M_1-b} \frac{d_j}{\gamma_j^{n+1}}, \ \ n \geq 0. \tag{2.10}$$

Remark: If the root of $D_1(z)$ in $|z| > 1$ are repeated, then the partial fractions has to be modified accordingly to acquire the probabilities. The same is elaboretly provided by Pradhan et al. [5].

2.7 Conclusions and future scopes

In this dissertation, we have studied basics of queueing system and different type of queues. A batch-service queue with size-dependent service time and general bulk service (a,b) rule has been elaborately discussed in which we have constructed transition probability matrix. Then, we have derived the probability generating function of queue-length. Finally, partical fraction has been incorporated to acquire the distribution in term of roots.
In future, the following can be incorporated in this model.

- Different vacation polocies can be included along with queue-length dependent vacation time.
- Also, server breakdown can be considered as this happens very often in industries.
- Other batch-service rules may be employed to this model which will lead to different set of new results.

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